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One-dimensionality and stability in legislative voting

Thomas Schwartz

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Abstract The widespread use in legislative studies of the one-dimensional model and its median-stability consequence raises a question: Do stability and one-dimensionality rest on evidence drawn from observed votes? They do not and cannot. I prove that every possible legislative history is compatible with a transitive majority preference (hence stability), and *ex*cept in very special circumstances with a cyclic majority preference (hence instability) as well: observed votes can never refute and almost never confirm stability. One-dimensionality fares worse: any legislative history is compatible with the one-dimensional model if it includes no two votes with overlapping pairs of alternatives, but otherwise, I show, it is almost certainly *in*compatible with the model, even in those rare cases that ensure transitivity. Voting evidence aside, the one-dimensional model is unduly restrictive, and arguments in its defense do not survive scrutiny.

Keywords Dimensionality · Single-peakedness · Legislatures · Voting

In legislative studies the one-dimensional model is so widely assumed, its median-stability consequence so prominently exploited, that a visiting scholar from Mars (or a terrestrial student new to the subject) would guess that stability and one-dimensionality rested on a bounty of evidence drawn from observed votes. Alas they do not, as a matter of logic they cannot, and as a matter of politics they had better not.

The one-dimensional model says that legislative alternatives can be ordered "left" to "right" so that each legislator likes them less the farther they lie to the left of his favorite or to its right. The model is prized for this consequence: At least one alternative—the median favorite—is stable, or unbeaten by any other, under majority rule. Model and consequence play starring roles in some influential but diverse books on Congress (Rohde 1991; Krehbiel 1998; Cox and McCubbins 1993, 2005) and in article after article on legislative

T. Schwartz (✉)
Department of Political Science, UCLA, 4289 Bunche Hall, Box 951472, Los Angeles, CA 90095-1472, USA
e-mail: tschwartz@polisci.ucla.edu

behavior. The left-to-right ordering of legislators according to Poole and Rosenthal (1997) NOMINATE scores, based on recorded votes, adds to the impression of empirical support.

After illustrating general conclusions with an example, then laying out the formal framework of discussion, I prove that no possible history of legislative votes can reveal instability, and except in very special circumstances that none can rule out instability either. Any history is compatible with the one-dimensional model if no two of its votes have overlapping pairs of alternatives. But if overlaps occur then violations of the model are not merely possible: they are nigh inevitable, even in those very special circumstances where votes ensure stability. I follow these results by examining arguments against and for the model and its consequence, concluding that one-dimensionality and stability are not only wanting in empirical support but downright implausible. The model is commonly violated even when Poole-Rosenthal NOMINATE scores reveal a single "dimension." Appendix A recasts that and other observations as methodological lessons.

1 Example

My results rest on an information gap: votes do not fully reveal preferences. To illustrate, suppose a three-member legislative chamber takes two votes, each pitting one alternative against another by majority rule, and we observe this:

At the first vote, z defeats x, with Reps. 2 and 3 voting for z, 1 for x.
At the second vote, y defeats z, with Reps. 1 and 2 voting for y, 3 for z.

Suppose too that Reps. 1, 2, and 3 are sincere (they vote for this rather than that only if they prefer this to that) and their preferences are transitive. Then we can infer the following, but no more, about those preferences:

Rep. 1 prefers both x and y to z.
Rep. 2 prefers y to z and both to x.
Rep. 3 prefers z to both x and y.

See the gap: we do not know 1 and 3's preferences between x and y.

Thanks to that gap, our information is compatible with two profiles of individual preference orderings, both shown in Fig. 1. The corresponding relations of majority preference are represented by arrows. One of them is transitive, the other cyclic. That makes y stable in the one case: no majority prefers anything to y. But no alternative is stable in the other.

We can portray preferences graphically by listing alternatives along the horizontal axis and interpreting the vertical axis as utility. The graphs in Fig. 1 capture the two profiles. In the first, each utility curve is single peaked: it is always rising or always falling or rising to a point and then falling. True, we can represent any preference with a single-peaked curve by suitably ordering alternatives on the horizontal axis. What is important is that some horizontal ordering makes all three curves single peaked at once. (Not every ordering does, but that does not matter.) In the second graph, the displayed ordering (xyz) does not make all three curves single peaked. Nor does any other ordering. We sum this up by saying of the two profiles themselves that the one is single peaked, the other not.

The one-dimensional model says that the prevailing profile is single peaked. So the first profile satisfies the model, but the observationally equivalent second one does not. In the first case, stable y is the median of legislators' favorites: no more than half of them have their favorites (or peaks) on the left of y or on its right. That is generally true: when the

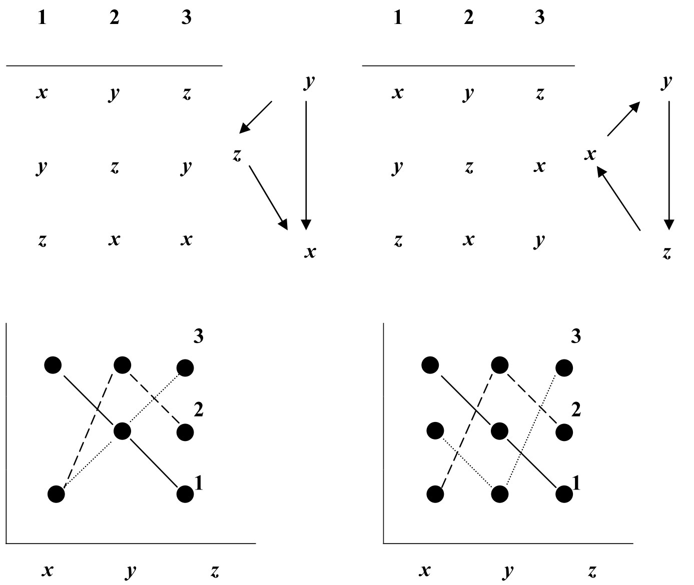

Fig. 1 Observationally equivalent profiles, with majority preferences and utility curves

model is satisfied, median favorites and they alone are stable. Then, too, majority preference is transitive, hence cycle free.

So at least in this simple case, one of two profiles is single peaked, a condition that ensures stability, whereas the other creates a cycle, thereby blocking stability and single peakedness, yet both are compatible with all one can observe from voting. In more elaborate cases, too, observed votes are almost always compatible with both stability and instability, though more rarely with single peakedness.

Objection: A third vote, between x and y, can close the information gap. If 1 votes for x and 3 for y, the first profile is the actual one. If 1 and 3 vote for x, the second is.

Reply: But then a defeated alternative (x) appears again. That is so rare that we may fairly assume it does not happen: it is hard enough to get anything on a real legislative agenda. Lawson (2000) counts only about two successful motions "to reconsider" per year in the U.S. Senate since WWII. And they had to reverse preceding votes, thereby changing some revealed preferences instead of revealing new preferences that might close gaps. Yes, a rejected amendment (a) to one bill (b_1) might get attached to another bill (b_2). But then the rejected *alternative* (outcome) would not be a but $b_1 + a$, and the new one would be $b_2 + a$. And yes, every successful bill defeats a default alternative, often the status quo, sometimes a zero appropriation. But a successful bill also alters the content of the default alternative: a later status quo or zero appropriation is a different alternative from the earlier, defeated one.

2 Formal framework

In substantive terms my discussion is about a legislative chamber of any size that chooses between any number of alternatives two at a time by majority rule. As in the example, assume that legislators are sincere, their preferences are transitive, no one abstains, there are no tie votes, and defeated alternatives do not reappear.

Formally, the advertised theorems treat of an integer n (number of legislators) and a set A (of legislative alternatives). Let $N = \{1, 2, \ldots, n\}$ and denote its members i, j, etc. Call a subset of N a *majority* if it has more than $n/2$ members, a *minority* if it has fewer than $n/2$. Call the members of A *alternatives* and denote them x, y, z, a, b, c, etc.

Individual preferences are represented by (strict linear) *orderings of* A, each a binary relation on A ($P \subseteq A^2$) that is asymmetric (never $xPyPx$), transitive ($xPyPz \Longrightarrow xPz$), and connected in A ($x \neq y \Longrightarrow xPy$ or yPx). Preference *profiles* are ordered n-tuples of orderings of A, denoted $\boldsymbol{P} = (P_1, \ldots, P_n)$, $\boldsymbol{P}' = (P_1', \ldots, P_n')$, etc. They determine relations of majority preference, defined:

$xM(\boldsymbol{P})y$ if and only if $\{i | xP_i y\}$ is a majority.

The findings to follow address four familiar conditions, three of them on $M(\boldsymbol{P})$:

TRANSITIVIY. If $xM(\boldsymbol{P})yM(\boldsymbol{P})z$ then $xM(\boldsymbol{P})z$.
ACYCLICITY. Never $x_1 M(\boldsymbol{P}) x_2 M(\boldsymbol{P}) \cdots M(\boldsymbol{P}) x_r M(\boldsymbol{P}) x_1$.
STABILITY. Every finite, nonempty subset of A has a member x such that $yM(\boldsymbol{P})x$ for no member y.

Obviously transitivity implies acyclicity, which is equivalent to stability. Black's (1948, 1958) famous theorem is that all three conditions follow from a fourth, this one on $\boldsymbol{P}$:

SINGLE PEAKEDNESS. There exists an ordering $\prec$ of A such that, for all i, x, y, z, if $xP_i y$ and if $z \prec y \prec x$ or $x \prec y \prec z$, then $xP_i z$.

In fact, it follows that the x mentioned by Stability is the (or a) median, relative to ordering $\prec$ and indexing population N, of P_i-best members. Single Peakedness is the most general and commonly cited version of one-dimensionality.[1]

To represent votes, define:

A *vote history* is any sequence $((W_1, x_1, y_1), \ldots, (W_h, x_h, y_h))$, $h \geq 1$, of ordered triples (W_i, x_i, y_i) in which W_i is a majority, $x_i \neq y_i$, and y_i is not among $x_{i+1}, \ldots, x_h, y_{i+1}, \ldots, y_h$.

Let $\boldsymbol{H} = ((W_1, x_1, y_1), \ldots, (W_h, x_h, y_h))$ denote an arbitrary history. Each (W_i, x_i, y_i) represents a vote in which x_i defeats y_i with W_i the winning majority: its members all vote for x_i, everyone else for y_i (so no one abstains). The reason y_i is banned from later occurrence in the history is that losers do not reappear.

As we saw in Sect. 1, two consequentially different preference profiles might be compatible with the same history of observed votes. Here is the general definition:

Profile $\boldsymbol{P}$ is *compatible with* history $\boldsymbol{H}$ if and only if, for all $k = 1, 2, \ldots, h$ and all i in N, $i \in W_k \Rightarrow x_k P_i y_k$ and $i \notin W_k \Rightarrow y_k P_i x_k$.

[1] The original version, proffered by Hotelling (1929) and popularized by Downs (1957), strengthens single peakedness by equating $\succ$-ordered A with the real line (ordered by magnitude) and making individual utility depend on distance from favorite points regardless of left or right direction. Single peakedness comes from Black (1948), who couched it in terms of utility curves. My more abstract formulation comes from Arrow (1963).

Infinite subsets of A need not have P_i-best members (legislators' favorites). So if A is infinite we must further assume that P_i-best members of A exist if we wish to deduce that a median of them is unbeaten *in A as a whole.*

Although the most general version of one-dimensionality, single peakedness is not the most general restriction on preference-combinations that ensures stability (acyclicity). Scores of weaker restrictions have been found over the years. The most general is Condorcet freedom (Schwartz 1986).

An $\boldsymbol{H}$-compatible $\boldsymbol{P}$ would include all preferences directly revealed by votes, so $x_k P_i y_k$ if i voted for $x_k (i \in W_k)$ but $y_k P_i x_k$ if not.

3 Two theorems: observed votes compatible with transitivity and cycles too

In Sect. 1 a two-vote history, although compatible with a cycle, was also compatible with transitivity. So is every history:

Theorem 1 *For every vote history $\boldsymbol{H}$, some profile $\boldsymbol{P}$ compatible with $\boldsymbol{H}$ makes $M(\boldsymbol{P})$ an ordering (hence transitive).*

The majority preference may in fact be cyclic, but votes can never reveal that fact: acyclicity is not falsifiable from votes. Appendix B has proofs of theorems.

The definition of "vote history" bans reconsideration: losing alternatives do not reappear. That in turn bans *directly revealed* cycles: if x defeats y and is then defeated by z, y does not reappear to defeat z. But those bans do not trivialize Theorem 1, because not every known majority preference is directly revealed. Each is composed of individual preferences, but some of those are inferred by transitivity from directly revealed preferences, and some of the latter are revealed by minority votes. Not that such things are ever enough to complete a cycle. But that has to be proved.

A famous kindred theorem is Szpilrajn's (1930): every (strict) partial ordering of a set (asymmetric, transitive) can be extended to an ordering (connected too). Some variants (Duggan 1999) show that we can start with less than a partial ordering (not cyclic but not transitive either). In effect Theorem 1 starts that way, with relations of revealed individual preference, up to n of them. Those relations are finite, something Szpilrajn does not assume. But Theorem 1 adds a wrinkle: we can not only extend those relations to full orderings but do it in a way that makes majority preference an ordering too.

According to Theorem 1 we can never observe a cycle. But can we ever observe acyclicity? Yes, in very special circumstances. Suppose $n = 3$, $A = \{x, y, z\}$, and we observe the two-vote history $((\{1, 2\}, x, y), (\{1, 2\}, z, x))$: Reps. 1 and 2 (a majority) vote for x against y, then for z against x. So they must prefer z to x to y and, therefore, z to y: majority preference is transitive.

But any vote history of realistic length and internal diversity allows cycles. Take a history comprising three or more votes, none unanimous (if only because we have deleted the unanimous ones). Suppose that either no winning alternative later loses (it keeps on winning or stops appearing, for a while or for good) or else that does happen but in at least one such case those legislators who vote for the winner and then against it (they are on the winning side both times) are a minority. That history must allow cycles:

Theorem 2 *Let $\boldsymbol{H}$ be any vote history in which $h \geq 3$, no W_i is N, and either $x_i = y_j$ for no i, j, or else $x_i = y_j$ for some i, j for which $W_i \cap W_j$ is a minority. Then some $\boldsymbol{P}$ compatible with $\boldsymbol{H}$ makes $M(\boldsymbol{P})$ cyclic.*

To sum up, observed votes are always compatible with a transitive majority preference, and except in very special circumstances with a cyclic one too.

4 Two more theorems: observed votes always compatible with single peakedness when issue pairs do not overlap, almost never when they do

Although every vote history is compatible with a transitive majority preference, transitivity does not ensure single peakedness; it is rather single peakedness that ensures transitivity. Is every history also compatible with single peakedness? No, but some are. It depends on whether *issue pairs*—the pairs of alternatives compared at different votes—ever overlap.

Histories in which they *never* overlap are perforce compatible with single peakedness—and, I may as well add, with the *violation* of single peakedness, too, so long as there exist at least three alternatives and one nonunanimous vote:

Theorem 3 *Let* $\boldsymbol{H}$ *be any vote history in which* $x_j \neq x_i \neq y_j$ *whenever* $i < j$. *Then* (a) *some* $\boldsymbol{H}$*-compatible profile is single peaked*, *and* (b) *provided* $|A| \geq 3$ *and some* W_j *is not* N, *some other* $\boldsymbol{H}$*-compatible profile is not single peaked.*

Given the trifling provisos of (b), when issue pairs do not overlap, votes tell us *nothing one way or the other* about one-dimensionality.

The ban on overlapping issue pairs is highly restrictive, of course. What if we allow some overlap? Then single peakedness is still a possibility, but now only barely. For $n = 3$ the two-vote history $((\{1, 2\}, x, y), (\{1, 2\}, z, x))$, with overlap x, allows and even guarantees single peakedness. But it represents a special case: besides only two votes, it has one majority $(\{1, 2\})$ on the winning side twice, both times all by itself.

By contrast, take any history that has some overlapping issue pairs and at least this much variation in winning sides: in some case of overlap, at least one legislator is on the winning side of either vote but not the other, and at least one is on the losing side both times. That is enough to block single peakedness:

Theorem 4 *Let* $\boldsymbol{H}$ *have components* (W_i, x_i, y_i) *and* (W_j, x_j, y_j) *that share an alternative, and suppose that* $W_i - W_j$, $W_j - W_i$, *and* $N - (W_i \cup W_j)$ *are nonempty. Then no profile compatible with* $\boldsymbol{H}$ *can be single peaked.*

To the importance of this theorem one might object that $\boldsymbol{H}$ can fall just a few votes short of compatibility with single peakedness. But in a legislature a few votes can be pivotal.

See what Theorem 4 adds to Theorem 2. When the hypothesis of Theorem 2 is satisfied, the observed votes merely *allow* a cycle, and with it a violation of single peakedness. But when the hypothesis of Theorem 4 is satisfied, the observed votes *compel* a violation of single peakedness (though not a cycle, thanks to Theorem 1). This can happen even in those rare circumstances that ensure transitivity. Suppose we have a two-vote history $((W_1, x_1, y_1), (W_2, x_2, y_2))$ with overlapping issue pairs and a majority, W, that is on the winning side both times. So $x_1 = x_2$ or $x_1 = y_2$, W is a subset of both W_1 and W_2, and transitivity is guaranteed. But so long as some legislator allies with W only on the first vote, another only on the second, and a third neither time, the hypothesis of Theorem 4 is satisfied and single peakedness is blocked.

5 Arguments against

Votes are not the whole story, of course, but when we examine some of the richer stories that stability and one-dimensionality rule out, those assumptions appear implausibly restrictive.

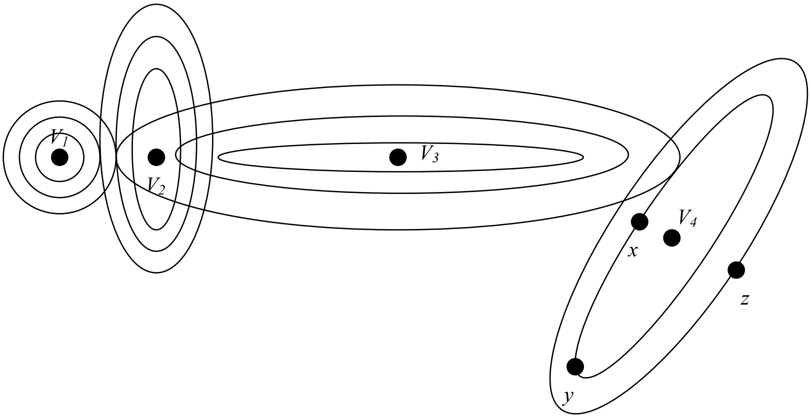

Fig. 2 Indifference maps

In showing this I shall traverse some familiar territory, but my route is partly new, and certain details of the topography are needed later, to report and refute the positive arguments of the next section.

By strengthening single peakedness one way, then relaxing it another, we arrive at the *general spatial model of voting*,[2] variously employed by critics as well as defenders of stability and one-dimensionality. The strengthening assumes that the feasible alternatives are all the points on a line, where each legislator has a favorite and likes other points less the farther they lie to its left or to its right. From now on let there be one median favorite (guaranteed if n is odd). Besides being unbeaten, Black (1948, 1958) showed, it beats every other point. *Proof*: Legislators whose favorite points are at or left (right) of the median are a majority, and they prefer it to every point on its right (left).

For relaxation we let the feasible alternatives be all the points in a Euclidean space of one or more dimensions, and if more than one we give every legislator i an *indifference map*. It consists of a favorite point V_i girt by *indifference contours*, infinitely many of them, so many so placed that every point lies on one and only one. Figure 2 is a partial picture of some indifference maps in two dimensions. Indifference contours are circles, ellipses, and other shapes, in general the boundaries of strictly convex sets. A legislator is indifferent between points (x, y) on the same contour and prefers points $(x, y$ again) on higher (inner) contours to ones (z) on lower (outer) contours. So Rep. 4 prefers y to z although z is closer to V_4. Only a legislator who, like Rep. 1, has circular (spheroidal) indifference contours always likes points less the farther they lie from his favorite, regardless of direction; only such a legislator, we say, has *Euclidean preferences*.

Now assume two or more dimensions, and take any line L. Besides a favorite point in the space, each legislator i has a favorite point on L—if not V_i then the point where one of i's indifference contours is tangent to L—and he likes other points less the farther they lie to its left or to its right. In Fig. 3, for example, Rep. 1 prefers V_1 to x to y to z to w, Rep. 2 prefers y (his favorite on L) to x to V_1 and also to z to w, etc. We may invoke Black's stability theorem and conclude that the median favorite on L beats every other point on L. But that result is now an *in*stability theorem. If a point x on L is not the median on L then x must

[2]From Black and Newing (1951), generalized by Davis and Hinich (1966) and Plott (1967).

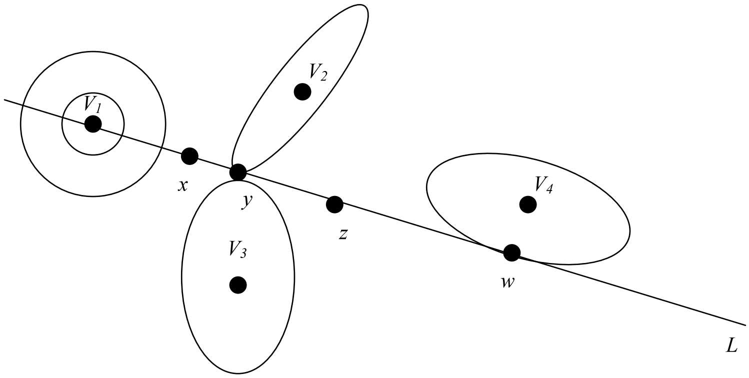

Fig. 3 Favorite points on any line

be unstable: the median on L beats x. So the only way x can be stable in the whole space is by being a *360° median*, a median on *every* line through x. That is an extraordinarily tough requirement: so perfectly balanced are legislators' indifference maps that, however much we rotate a line at x, we shall never find the favorite (tangency) points of more than $n/2$ legislators on either side of x.[3]

Of course one might question the general spatial model, especially its assumption that the feasible alternatives are the uncountable infinitude of points in a Euclidean space; I would. But if we start with that model we find that Black's theorem is two edged: precisely by ensuring stability in one dimension it all but precludes stability in more dimensions. This does not gainsay one-dimensionality. It does show that the number one is critical: once we assume two or more dimensions we not only lose the guarantee of stability but buy a near guarantee of instability. For that reason the one-dimensional model cannot be rationalized, with an epistemological shrug, as a simplification or abstraction or reduced form, got by uncluttering one explanatory factor. Its champions must argue that the number of dimensions really is exactly one, or that a plurality of dimensions can be effectively reduced to one. I examine some such arguments in the next section.

For a less fanciful way to appreciate how restrictive acyclicity and single-peakedness are, imagine two legislative measures, a and b, each designed to benefit some minority at every one else's expense. Suppose the two minorities share no members but together make a majority, and there are mutual gains from trade: both minorities prefer the passage of both measures (ab) to the defeat of both ($\overline{ab}$). Then legislators must have the following preferences:

[3] Assuming the general spatial model with two or more dimensions, the near impossibility of a stable point—the extreme severity of a necessary condition for stability—was discovered by Plott (1967). His theorem is more dramatic but harder to prove than mine about a 360° median, which is more reminiscent of Cox's (1987). Its equivalence to Black's old theorem is new.

Assuming that x is the favorite point of at most one legislator, Plott's necessary condition for the stability of x is *pairwise symmetry*: legislators whose favorite points are *not* x can be paired one-to-one so that each has an indifference contour convex and tangent to his mate's at x (i.e., x is on their contract curve). It is not too hard to see that if x is *not* pairwise symmetric but is the median tangency point on some line L then an arbitrarily small rotation of L at x in some plane destroys x's medianhood on the rotated line, proving thereby x's instability (since it cannot be a 360° median) and therewith Plott's theorem.

Minority 1 prefers $a\bar{b}$ to ab to $\overline{ab}$ to $\bar{a}b$.
Minority 2 prefers $\bar{a}b$ to ab to $\overline{ab}$ to $a\bar{b}$.
Everyone else prefers $\overline{ab}$ to $a\bar{b}$ and $\bar{a}b$ and both to ab.

These are exactly the preferences needed for a vote trade. If both measures are voted on independently, both fail: the outcome is $\overline{ab}$, the status quo. But if the two minorities trade votes, Minority 1 supporting b in return for 2's support of a, then both measures pass: the outcome is ab, which the majority of minorities prefers to $\overline{ab}$. But this apparently common scenario has created a cycle: majorities prefer ab to $\overline{ab}$ to $a\bar{b}$ back to ab—also ab to $\overline{ab}$ to $\bar{a}b$ to ab.

Explicit vote trades are not needed. We can think of ab as a single bill written to please both minorities. If we like we can imagine that this bill contains, besides a and b, some broadly appealing provisions not decomposable into minority benefits: let $\overline{ab}$ be the package of those provisions rather than the status quo, and suppose most legislators prefer $\overline{ab}$ to the actual status quo. We can also allow shared members: they prefer ab to $a\bar{b}$ and $\bar{a}b$, and all three to $\overline{ab}$. And instead of two large minorities we can of course concoct similar examples with three or more smaller minorities. Any bill or package of bills fits the pattern if it benefits a majority of minorities, each at everyone else's expense.[4] I think you will agree that a great deal of legislation is like that.

This story opens the door to evidence of a sort, but by Theorem 1 it cannot consist solely of votes. Shared members aside, the preferences that make up the cycle include Minority 1's preference for $\bar{b}$ over b regardless of a. But if the two minorities trade votes then Minority 1 will not *reveal* that preference: it will vote for b against $\bar{b}$. Or if ab is a single bill then Minority 1 would, if similarly cooperative, support ab against any amendment to strike b. Evidence for the cycle can be found, not by observing votes, but by unraveling ab into component measures, then following the money to see which constituencies and other groups tied to Reps. $1, 2, \ldots, n$ stand to benefit from which of those measures.

6 Arguments for

Naturally there are arguments on the other side, not all of them explicit: the one-dimensional model is often assumed, rarely defended. But the arguments below rest on familiar premises and probably account for much of the model's appeal.

One argument allows multiple dimensions but hypothesizes that they amount to one because *monotonically related*: if Rep. i's favorite point in one dimension is right of Rep. j's then that is true in all dimensions. Equivalently: each legislator is ideologically consistent. Enelow and Hinich (1984: 38ff.) make much the same point, stochastically qualified, by hypothesizing an underlying "predictive dimension." It is supposed to follow that we can treat any of those dimensions as the sole one and that the vector of dimensional medians is stable.

But suppose $n = 3$, the dimensions are two, and preferences are Euclidean. Then the fancied monotonicity is consistent with both graphs in Fig. 4. The first is linear, and median V_2 beats every other point in the space. But linearity is an extraordinarily special case of monotonicity (itself a strong assumption). The second graph is much more likely. There V_2, the vector of dimensional medians, is not stable: x is closer than V_2 to V_1 and V_3, so Reps. 1 and 3 (a majority) prefer x to V_2. (Not that x is stable either: y beats it.)

[4]The connection between majorities of minorities and instability was found by Downs (1957: 54–60), then successively generalized by Kadane (1972), Bernholz (1973), Schwartz (1977), and Schwartz (1981).

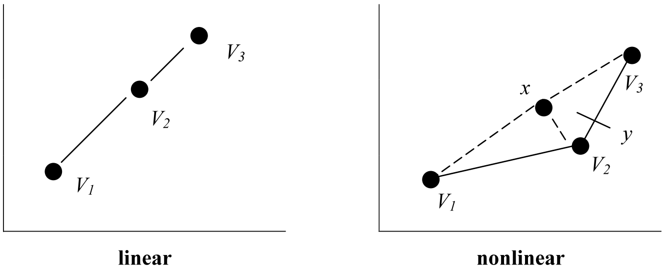

Fig. 4 Monotonically related dimensions

The best-known defense of the one-dimensional model begins with a concession: let there be several dimensions, representing as many policy issues. But suppose legislators vote one dimension at a time. Then we may as well assume one-dimensionality because every contest among rival measures fits that assumption. To justify the hypothesis that legislators vote one dimension at a time, Shepsle (1979) and Shepsle and Weingast (1981), focusing on the U.S. House of Representatives, famously contend that the jurisdiction of each legislative committee is one-dimensional, that every floor contest among rival measures pits a committee report against amended variants, and that a germaneness requirement squeezes all recognizable amendments into the same one-dimensional committee jurisdiction as the reported bill. As these authors point out, we must assume as well that committee jurisdictions are not linked by vote trades or complementary preferences.

But how plausible is it that committee jurisdictions fit single dimensions? Typical bills are nothing like points on a line. They are book-length compendia of services, handouts, penalties, contingencies, mandates, restrictions, and procedures, each burdening or benefiting this or that group, often a minority. General-interest legislation (e.g., national defense) is larded with minority benefits (bases, procurements). Even when not explicitly porcine, domestic spending is based on formulas finely tuneable to favor one or another minority (school construction, water and sewage plants, farm subsidies). The U.S. tax code would take up a pamphlet, not a small library, but for countless minority subsidies and exemptions. Our package measure *ab* was if anything a gross oversimplification yet already complex enough to block one-dimensionality and even stability. To add some substance to that schematic story, interpret *a* as agricultural price supports, *b* as food stamps, and Minorities 1 and 2 as representatives of rural and urban Congressional districts, and you have the Agriculture Act of 1977. Well aware of the instability, floor leaders protected *ab* against majority-preferred amendments by lining up counter-amendments to threaten their sponsors (McCubbins and Schwartz 1988).

The one-dimensional model appears to gain plausibility from the way we talk about politicians. We have no trouble placing them in left-to-right order, and for Congress ADA scores do that with a certain precision. But all this shows is that each legislator has an ideological average, reckoned from his votes on conspicuously ideological issues. That is necessarily true, however he and others vote.

A particularly interesting source of potential support for the one-dimensional model is the successful left-to-right ordering of legislators according to the NOMINATE scores of Poole and Rosenthal (1985, 1997). Their method fits a single scale, or factor, to the bulk of congressional voting (not quite all of it). It thereby encodes the probability of any two legislators voting alike using indices assigned to single legislators rather than pairs of them: the closer your index is to mine, the more frequently we vote the same way. But for all its predictive power and beauty the NOMINATE ordering is not necessarily the left-to-right ordering required by single peakedness (the one-dimensional model).

To see why, suppose x defeats y, then z defeats x, with majority *MAJ* voting for x then z, minority *MIN* voting the opposite way both times, and swing legislator *SW* voting with *MIN* for y then with *MAJ* for z. Here are the revealed preferences:

MAJ prefers z to x to y.
SW prefers z and y to x.
MIN prefers y to x to z.

Obviously majority preference is transitive. By the Poole-Rosenthal method, the first vote divides N into *MAJ* and $MIN + SW$, the second into $MAJ + SW$ and *MIN*, together requiring the left-to-right ordering $MAJ - SW - MIN$—or the reverse. Even so, single peakedness is already violated. If we order the three alternatives xyz (or the reverse), then *MAJ* ends up with a preference curve that is not single peaked, likewise *MIN* with yxz, and likewise *SW* with xzy.

The lesson is quite general:—To order legislators from left to right is one thing, often an easy thing. To order both legislators and alternatives, as required by single peakedness, is something else, often an impossibility. In the example the ordering of legislators was simple and uncontroversial, but we could not mesh it with any ordering of alternatives. Why has this not been appreciated? Because, I believe, most scholars who work with legislative votes do not look for overlaps between issue pairs (hard to discern from roll-call reports), but without them single peakedness cannot be falsified, says Theorem 3.

7 Conclusion

Its simplicity and fruitfulness make the one-dimensional model valuable as a source of examples of what *can* happen and of insights that invite generalization. Take the celebrated "setter" model of Romer and Rosenthal (1978).[5] It starts with Hotelling's (1929) strong version of one-dimensionality: alternatives are all the points on a line, and each legislator likes them less the farther they lie from his favorite, regardless of left or right direction. If proposals are considered under a closed rule (no amendments), and if the default alternative lies far to one side of the median favorite point, then a successful proposal can lie almost as far to the other side. Here one-dimensionality makes things vivid, but it is not essential to the lessons (1) that the less popular the default alternative (by any or many measures), the greater the range of legislation that can pass under a closed rule, and (2) that a take-it-or-leave-it agenda setter can be surprisingly influential although someone else has the last word. More dimensions would, if anything, magnify these lessons, only the graphics would be harder to visualize and explain.

But as a general hypothesis about legislative behavior, neither the one-dimensional model nor its stability consequence rests, or can rest, on the evidence of votes. Observed votes can never refute stability (or acyclicity), can almost never confirm stability, can never confirm or refute one-dimensionality (or single peakedness) if issue pairs do not overlap at all, but almost always do refute one-dimensionality if they overlap even a bit. Moreover, to affirm stability or one-dimensionality within any legislative context, one must be prepared to affirm that successful legislation in that context never benefits a majority of minorities, each at everyone else's expense.

[5] It is too bad those authors do not have Celtic names, else we could speak of the Irish setter model.

It is no good protesting that sound science relies on "maintained assumptions," not tested and confirmed every time they are used. Stability is not even testable from observed votes except in very special circumstances, and the one-dimensional model, although more often testable, is most often refuted. It is no good protesting that some assumptions are, like economic rationality, hard to test but plausible enough to wrap inside readily tested models. Stability and one-dimensionality are not that plausible; they are not plausible at all in a world, such as the real one, where legislation sometimes benefits majorities of minorities. It is no good protesting that the one-dimensional model is not essential anyway to the observational generalizations explained thereby. Empirical explananda never necessitate their theoretical explanantia. It is no good protesting that simple, fruitful models have to fit the evidence only approximately and then only after outliers have been purged. What we found is not that stability and one-dimensionality fit observed votes poorly but that stability fits too well—it cannot be falsified—and one-dimensionality hardly at all. And it is no good protesting that models based on one-dimensionality are nothing more than elaborate examples. Then one should have said so and said what they exemplify.

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Appendix A: Methodological warnings

These eight warnings are practical lessons from social choice theory for legislative studies. Each treats only of common or typical cases, ignoring possible but *outré* exceptions.

1. Do not try to infer stability or instability, cycles or acyclicity, from a vote history that reports votes between alternatives but says nothing about the content of those alternatives.
2. When a legislative history reports overlapping issue pairs, expect a violation of single peakedness.
3. When it does not, expect to find no violation but also no positive evidence of single peakedness.
4. When a legislative alternative would benefit a majority of minorities, each at everyone else's expense, expect to find that it is unstable: a majority prefer another alternative to it.
5. In the multi-dimensional spatial model of voting, even if preferences are Euclidean and the dimensions are *monotonically* related (legislators' ideal points line up the same way in all dimensions), do not expect stability or single peakedness to hold. Those properties are satisfied only if the dimensions are *linearly* related. Linearity is a very special case of monotonicity.
6. When a germaneness rule limits legislative voting on a bill and amended variants to a single "subject" or committee jurisdiction, that obviously limits the number of dimensions of the alternatives on the floor. But do not expect it to reduce that number to one. And unless it does, do not expect it to ensure stability (or acyclicity) among those alternatives.

7. Shepsle and Weingast cited germaneness only as an example of a procedural rule that can help secure stability—as a “structure” that can “induce equilibrium.” But whether procedural rules have that effect depends on what kind of “stability” one has in mind. The creature of majority-preference cycles, an alternative’s *latent instability* is the preference by a majority for another alternative; its *manifest instability* is its actual rejection in a vote in favor of another alternative. The 1977 Agriculture Act was latently unstable, as I explained, but it was manifestly stable: it endured for four years, and in outline it passed again in 1981. Obviously procedural rules contribute to manifest stability, e.g., by making it hard to reconsider a bill once passed. But single peakedness and Theorems 1 and 2 have to do only with latent stability, which cycles block. Do not confuse the two.
8. Sound methods of dimensional analysis for roll-call votes, such as Poole-Rosenthal NOMINATE scoring, demonstrably achieve their advertised purpose of matching proximity between legislators with the probability of their voting alike. But do not expect the dimensions found thereby to serve other purposes (Koford 1989), and when those dimensions turn out to be one in number, do not infer single peakedness.

Appendix B: Proofs

To prove Theorems 1–4 we shall need four more definitions and three lemmata. Define:

z *defeats* w in $\boldsymbol{H}$ iff $(z, w) = (x_i, y_i)$ for some $i \le h$.
D_z^H = the least set containing z and containing everything defeated in $\boldsymbol{H}$ by any member.

Lemma 1 *Each thing is defeated in* $\boldsymbol{H}$ *by at most one thing.*

Proof Otherwise, some y_i would be defeated in $\boldsymbol{H}$ by x_i, then by some other x_j, so that $y_i = y_j$ though $i < j$, contrary to the definition of “history.” □

Lemma 2 *In* $\boldsymbol{H}$, *no member of* D_Z^H *defeats any nonmember, and no nonmember defeats any member but* z.

Proof Everything defeated by a member is itself a member, so a member cannot defeat a nonmember. Also every member but z is defeated by another member, so by Lemma 1 it cannot be defeated by a nonmember. □

Now define:

z is *compared with* w in $\boldsymbol{H}$ iff z defeats w or w defeats z in $\boldsymbol{H}$.
If P is an ordering of A and $X \subseteq A$, let $[XP]$ be the permutation of P got by moving the P-ordering of X *above* that of $A - X$, and $[PX]$ the permutation got by moving the P-ordering of X *below* that of $A - X$. Further let $[xP] = [\{x\}P]$ and $[Px] = [P\{x\}]$.

Lemma 3 *Suppose* $\boldsymbol{P}$ *is compatible with* $\boldsymbol{H}$. *Take any* z *and* $i \le h$. *If nothing below* z *in* P_i *defeats* z, *then permuting* P_i *to* $[P_i D_z^H]$ *preserves* $\boldsymbol{H}$*-compatibility. And if nothing above* z *in* P_i *defeats* z, *then permuting* P_i *to* $[D_z^H P_i]$ *preserves* $\boldsymbol{H}$*-compatibility.*

Proof By construction and Lemma 2, the relative positions of any two alternatives in P_i are changed by the given permutations only if those alternatives are never compared in $\boldsymbol{H}$. Therefore, such changes must preserve $\boldsymbol{H}$-compatibility. □

Theorem 1 *For every $\boldsymbol{H}$, some $\boldsymbol{H}$-compatible $\boldsymbol{P}$ makes $M(\boldsymbol{P})$ an ordering of A.*

Proof By induction on h.

Suppose $h = 1$, so that $\boldsymbol{H} = ((W_1, x_1, y_1))$. Let R^1 be any ordering of A with x_1 at top, and R_1 the same but with x_1 at bottom. Let $P_i = R^1$ whenever $i \in W_1$, and $P_i = R_1$ whenever $i \notin W_1$. Then obviously $\boldsymbol{P} = (P_1, \ldots, P_n)$ is $\boldsymbol{H}$-compatible. And because W_1 is a majority, $M(\boldsymbol{P})$ is the ordering R^1.

Now suppose $h > 1$ and the theorem holds for $\boldsymbol{H}' = ((W_1, x_1, y_1), \ldots, ((W_{h-1}, x_{h-1}, y_{h-1}))$: some $\boldsymbol{P}$ compatible with $\boldsymbol{H}'$ makes $M(\boldsymbol{P})$ an ordering. Let:

$$P_i^* = \begin{cases} [D_{x_h}^{H'} P_i] & \text{if } i \in W_h \\ [P_i D_{x_h}^{H'}] & \text{otherwise}, \ i = 1, 2, \ldots, n. \end{cases}$$

By definition of "history," x_h differs from every y_i, so nothing defeats x_h in $\boldsymbol{H}'$. Therefore, $\boldsymbol{P}^* = (P_1^*, \ldots, P_n^*)$ is, like $\boldsymbol{P}$, $\boldsymbol{H}'$-compatible by Lemma 2. Also nothing defeats y_h in $\boldsymbol{H}'$, so $y_h \notin D_{x_h}^{H'}$. Consequently, in every P_i^*, x_h is above y_h if $i \in W_h$ but below y_h otherwise. Hence $\boldsymbol{P}^*$ is $\boldsymbol{H}$-compatible too. Moreover, because $M(\boldsymbol{P})$ is an ordering and W_h is a majority, $M(\boldsymbol{P}^*)$ is the permuted ordering $[D_{x_h}^{H'} M(\boldsymbol{P})]$. □

Theorem 2 *Let $\boldsymbol{H}$ be any history with $h \geq 3$, $W_i \neq N \forall i$, and either $x_i = x_j$ for no i, j, or else $x_j = y_k$ for some j, k for which $W_j \cap W_k$ is a minority. Then some $\boldsymbol{P}$ compatible with $\boldsymbol{H}$ makes $M(\boldsymbol{P})$ cyclic.*

Proof Some $\boldsymbol{P}$ is compatible with $\boldsymbol{H}$ by Theorem 1. Two main cases.

Case 1. $x_i = y_j$ for no i, j: no x_i is defeated in $\boldsymbol{H}$. Two subcases.

Subcase 1.1. $x_i = x_j$ for all j. Then x_1 defeats y_1, y_2, y_3. To aid readability, rewrite x_1, y_1, y_2, y_3 as x, a, b, c, respectively. So x defeats a, b, and c.

In each P_i, x is either above a, b, and c, below all three, or above only a, only b, only c, only a and b, only a and c, or only b and c. But the cases where x is above only a or above only a and c are all cases where x is below b and above a. Two other pairs of cases can be likewise combined. As a result, N can be partitioned into these five sets:

$$S_c^a = \{i \,|\, x \text{ is below } a \text{ and above } c \text{ in } P_i\},$$

$$S_a^b = \{i \,|\, x \text{ is below } b \text{ and above } a \text{ in } P_i\},$$

$$S_b^c = \{i \,|\, x \text{ is below } c \text{ and above } b \text{ in } P_i\},$$

$$S^{abc} = \{i \,|\, x \text{ is below } a, b, c \text{ in } P_i\},$$

$$S_{abc} = \{i \,|\, x \text{ is above } a, b, c \text{ in } P_i\}.$$

Because x defeats a (in $\boldsymbol{H}$), there exists a majority W with x above a in P_i for all i in W. But x is below a in P_i for every i in S_c^a. Therefore, S_c^a is a minority. So likewise are S_a^b and $S_b^c \cup S^{abc}$.

Now partition S_{abc} into subsets S_{abc}^1, S_{abc}^2, and S_{abc}^3 (possibly empty in one or more cases) so that $S_{abc}^1 \cup S_c^a$, $S_{abc}^2 \cup S_a^b$, and $S_{abc}^3 \cup S_b^c \cup S^{abc}$ are still minorities. But these three sets partition N. So any two of them make a majority.

Because x defeats a, nothing else can defeat a by Lemma 1, and because $x = x_1, a = y_1$ can defeat nothing. So a is comparable only to x. Therefore, $\boldsymbol{H}$-compatibility is preserved if we move a to the top of any P_i in which aP_ix or to the bottom of any P_i in which xP_ia. Likewise b and c. Hence, the following permutations preserve $\boldsymbol{H}$-compatibility:

$$
\begin{aligned}
&P_i \text{ to } P_i' = [[aP_i]\,c] \quad \text{if } i \in S_c^a,\\
&P_i \text{ to } P_i' = [[bP_i]\,a] \quad \text{if } i \in S_a^b,\\
&P_i \text{ to } P_i' = [[cP_i]\,b] \quad \text{if } i \in S_b^c,\\
&P_i \text{ to } P_i' = [c\,[a\,[bP_i]]] \quad \text{if } i \in S^{abc},\\
&P_i \text{ to } P_i' = [[[P_ia]\,b]\,c] \quad \text{if } i \in S_{abc}^1,\\
&P_i \text{ to } P_i' = [[[P_ib]\,c]\,a] \quad \text{if } i \in S_{abc}^2,\\
&P_i \text{ to } P_i' = [[[P_ic]\,a]\,b] \quad \text{if } i \in S_{abc}^3.
\end{aligned}
$$

This follows:

$$
\begin{aligned}
&aP_i'bP_i'c \quad \text{if } i \in S_{abc}^1 \cup S_c^a,\\
&bP_i'cP_i'a \quad \text{if } i \in S_{abc}^2 \cup S_a^b, \quad \text{and}\\
&cP_i'aP_i'b \quad \text{if } i \in S_{abc}^3 \cup S_b^c \cup S^{abc}.
\end{aligned}
$$

Because any two of those subsets make a majority we have $aM(\boldsymbol{P}')bM(\boldsymbol{P}')cM(\boldsymbol{P}')a$. So $\boldsymbol{H}$-compatible $\boldsymbol{P}'$ makes $M(\boldsymbol{P}')$ cyclic.

Subcase 1.2. $x_1 \neq x_j$ for some j. Then there must exit $j < h$ such that x_j occurs in no triple beyond (W_j, x_j, y_j). Because $W_j \neq N$, we can partition W_j into minorities W_j^1 and W_j^2. To aid readability, rewrite x_j, y_j, x_{j+1} as x, y, z, respectively.

By hypothesis of the case, nothing defeats z (in $\boldsymbol{H}$). So compatibility is preserved, by Lemma 3, if we permute any P_i to $[D_z^H P_i]$ or $[P_i D_z^H]$. Again by hypothesis of the case, because x defeats y, y cannot defeat anything, and by Lemma 1 nothing else defeats y. So y is comparable to naught but x. However, y is above x in every P_i with $i \in N - W_j$. So compatibility is preserved when any such P_i is permuted to $[yP_i]$. Hence, the following permutations preserve compatibility:

$$
\begin{aligned}
&P_i \text{ to } P_i' = \left[D_z^H P_i\right] \quad \text{if } i \in W_j^1,\\
&P_i \text{ to } P_i' = \left[P_i D_z^H\right] \quad \text{if } i \in W_j^2, \quad \text{and}\\
&P_i \text{ to } P_i' = \left[y\left[D_z^H P_i\right]\right] \quad \text{if } i \in N - W.
\end{aligned}
$$

But now we have:

$$
\begin{aligned}
&zP_i'xP_i'y \quad \text{for all } i \in W_j^1,\\
&xP_i'yP_i'z \quad \text{for all } i \in W_j^2, \quad \text{and}\\
&yP_i'zP_i'x \quad \text{for all } i \in N - W.
\end{aligned}
$$

Because $\{W_j^1, W_j^2, N - W\}$ is a partition of N into three minorities, any two of which make a majority, we further have $xM(\boldsymbol{P}')yM(\boldsymbol{P}')zM(\boldsymbol{P}')x$—a cycle.

Case 2. $x_j = y_k$ for some j, k for which $W_j \cap W_k$ is a minority. It follows from the definition of "history" that $j < k$. Let us relable x_j, y_j, x_k as x, y, z, respectively. So x defeats y (at the jth triple) and z defeats x (at the kth).

Because $\boldsymbol{P}$ is $\boldsymbol{H}$-compatible, we have, for all i:

$$xP_iy \text{ and } xP_iz \quad \text{if } i \in W_j - W_k, \tag{1}$$

$$zP_ixP_iy \quad \text{if } i \in W_j \cap W_k, \tag{2}$$

$$yP_ix \text{ and } zP_ix \quad \text{if } i \in W_k - W_j, \quad \text{and} \tag{3}$$

$$yP_ixP_iz \quad \text{if } i \in N - (W_j \cup W_k). \tag{4}$$

Because $x = y_k$ but an alternative once defeated does not recur in any triple, every member of B_x^H besides x is defeated before the kth triple (it is defeated in $((W_1, x_1, y_1), \ldots, (W_{k-1}, x_{k-1}, y_{k-1})))$ and thus cannot appear in the kth triple, (W_k, x_k, y_k). So $z = y_k \notin B_x^H$. Likewise $z \notin B_y^H$.

But because x defeats y, $y \in B_x^H$. So if we permute P_i to $[B_x^H P_i]$ whenever $i \in W_j - W_k$, we have x above y above z in the permuted ordering. And because, by (1), x was already above z, which defeats x, the permutation preserves $\boldsymbol{H}$-compatibility thanks to Lemma 3.

Similarly, if we permute P_i to $[B_x^H P_i]$ whenever $i \in W_k - W_j$, we have y above z above x by (3), and $\boldsymbol{H}$-compatibility is preserved by Lemma 3.

Now change $\boldsymbol{P}$ to $\boldsymbol{P}'$ by effecting those two permutations. Then $\boldsymbol{P}'$ is $\boldsymbol{H}$-compatible. But now we have:

$$xP_i'y \quad \text{for every } i \in W_j,$$

$$zP_i'x \quad \text{for every } i \in W_k, \quad \text{and}$$

$$yP_i'z \quad \text{for every } i \in N - (W_j \cap W_k).$$

But $W_j \cap W_k$ is a minority by hypothesis of the case, making $N - (W_j \cap W_k)$ a majority, as are W_j and W_k. Hence $xM(\boldsymbol{P}')yM(\boldsymbol{P}')zM(\boldsymbol{P}')x$, another cycle. □

Theorem 3 *Let $\boldsymbol{H}$ be such that $x_j \neq x_i \neq y_j$ whenever $i < j$. Then* (a) *some $\boldsymbol{H}$-compatible $\boldsymbol{P}$ is single peaked*, *and* (b) *provided $|A| \geq 3$ and $W_j \neq N$ for some j*, *some $\boldsymbol{H}$-compatible $\boldsymbol{P}$ is not single peaked.*

Proof (a) Let R be any ordering of $N - \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$. Construct $\boldsymbol{P}$ so that, for all i, $x_kP_iy_k$ if $i \in W_k$, $y_kP_ix_k$ if $i \notin W_k$, and P_i has x_1 and y_1 above x_2 and y_2, the latter above x_3 and y_3, etc., with R below all of them. Then $\boldsymbol{P}$ is compatible with $\boldsymbol{H}$. Now construct $\prec$ so that

$$x_h \prec x_{h-1} \prec \cdots \prec x_1 \prec y_1 \prec y_2 \prec \cdots \prec y_h$$

with R again below all the x_j, y_j. Then

$$aP_ib \Rightarrow aP_ic \quad \text{whenever} \quad c \prec b \prec a \text{ or } a \prec b \prec c.$$

So $\boldsymbol{P}$ is single peaked.

(b) By hypothesis, nothing defeats any x_i in $\boldsymbol{H}$. Because $W_j \neq N$ we can partition W_j into two minorities, W_j^1 and W_j^2. Let $x = x_j$, $y = y_j$. If $h > 1$ let $z = x_i$ for some $i \neq j$, and if $h = 1$ let $z \in A - \{x, y\}$ (possible because $|A| \geq 3$). Either way, nothing defeats z in $\boldsymbol{H}$, and the argument of Subcase 1.2 in the previous proof goes through. So some $\boldsymbol{P}$ compatible with $\boldsymbol{H}$ makes $M(\boldsymbol{P})$ cyclic, hence not single peaked. □

Theorem 4 *If components* (W_j, x_j, y_j) *and* (W_k, x_k, y_k) *of* $\boldsymbol{H}$ *share an alternative, and if* $W_j - W_k$, $W_k - W_j$, *and* $N - (W_j \cup W_k)$ *are nonempty, and if* $\boldsymbol{P}$ *is compatible with* $\boldsymbol{H}$, *then* $\boldsymbol{P}$ *cannot be single peaked.*

Proof Let $q \in W_j - W_k, r \in W_k - W_j$, and $s \in N - (W_j \cap W_k)$. Because W_j and W_k are majorities, we may further let $t \in W_j \cap W_k$. Because $W_j - W_k \neq \phi$, $j \neq k$; say $j < k$. Then y_j cannot occur in (x_k, y_k). So x_j does, and there are two cases to consider.

Case 1. $x_j = x_k$. But $x_j P_i y_j$ for all $i \in W_j$ whereas $y_j P_i x_j$ for all $i \in N - W_j$. Also $x_k = x_j P_i y_k$ for all $i \in W_k$ whereas $y_k P_i x_j = x_k$ for all $i \in N - W_k$. So:

$$y_k P_q x_j P_q y_j, \quad x_j P_t y_j \& x_j P_t y_k, \quad y_j P_r x_j P_r y_k, \quad \text{and} \quad y_j P_s x_j \& y_k P_s x_j.$$

But then $\boldsymbol{P}$ flouts single peakedness. For example, if the $\prec$-ordering has $x_j \prec y_j \prec y_k$ then obviously P_q does not fit. It is a routine matter to check the other possible $\prec$-orderings.

Case 2. $x_j = y_k$. As before, $x_j P_i y_j$ for all $i \in W_j$ whereas $y_j P_i x_j$ for all $i \in N - W_j$. But now $x_k P_i x_j = y_k$ for all $i \in W_k$ whereas $y_k = x_j P_i x_k$ for all $i \in N - W_k$. So:

$$x_j P_q y_j \& x_j P_q x_k, \quad x_k P_t x_j P_t y_j, \quad x_k P_r x_j \& y_j P_r x_j, \quad \text{and} \quad y_j P_s x_j P_s x_k.$$

This again is enough to bar some $\boldsymbol{P}$-ordering from fitting any given $\prec$-ordering. □

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